THE EXISTENCE OF A NEAR-UNANIMITY TERM IN A FINITE ALGEBRA IS DECIDABLE

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Abstract. We prove that it is decidable of a finite algebra whether it has a near-unanimity term operation, which settles a ten-year-old problem. As a consequence, it is decidable of a finite algebra in a congruence distributive variety whether it admits a natural duality.

Introduction

We call an operation f a *near-unanimity operation* if it satisfies the identity

$$f(y, x, \dots, x) \approx f(x, y, x, \dots, x) \approx \dots \approx f(x, \dots, x, y) \approx x.$$

Near-unanimity term operations come up naturally in the study of algebras. For example, if an algebra of finite signature has a near-unanimity term operation, then it has a finite base of equations. It was asked in [1] whether having a near-unanimity term operation is a decidable property of a finite algebra. This problem, called the *near-unanimity problem*, is intimately linked to the problem of deciding whether a finite algebra admits a natural duality, called the *natural duality problem*. B. Davey and H. Werner proved in [2] that if an algebra has a near-unanimity term operation then it admits a natural duality. In the case of algebras in a congruence distributive variety the converse was proved in [1]. This result, called the *near-unanimity obstacle theorem*, states that an algebra in a congruence distributive variety admits a natural duality if and only if it has a near-unanimity term operation. Note, that it is easy to decide of a finite algebra whether it lies in a congruence distributive variety by searching for Jónsson terms.

Clearly, an algebra $\mathbf{A}$ has a near-unanimity term operation f if and only the equations

$$(*) \qquad f(y, x, \dots, x) = f(x, y, x, \dots, x) = \dots = f(x, \dots, x, y) = x$$

hold for the generator elements x, y of the two-generated free algebra in the variety generated by $\mathbf{A}$. Probably this observation motivated R. McKenzie's unpublished result [8] where he proves that it is undecidable of a finite algebra $\mathbf{A}$ and two fixed elements $x, y \in A$ whether $\mathbf{A}$ has a term operation that behaves as a near-unanimity operation on $\{x, y\}$. Later, this result was

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slightly extended in [7], where it is proved that given a finite algebra $\mathbf{A}$ and two fixed elements $a, b \in A$, it is undecidable if $\mathbf{A}$ has a term operation that behaves as a near-unanimity operation on $A \setminus \{a, b\}$, that is, equations $(*)$ hold for all pairs $x, y \in A \setminus \{a, b\}$.

In this paper we show that the near-unanimity problem is decidable, see Theorem 17, which is a rather surprising development after the negative partial results. Since there are only finitely many algebras on a fixed n-element set whose basic operations are at most r-ary, there must exist a recursive function $N(n, r)$ that puts an upper limit on the minimum arity of near-unanimity term operations for those algebras that have one. Consequently, given an algebra $\mathbf{A}$ whose operations are at most r-ary, one can decide the near-unanimity problem by simply calculating all at most $N(|A|, r)$-ary terms and checking if one of them yields a near-unanimity operation. If no such is found, then $\mathbf{A}$ has no near-unanimity term operation. We know that such recursive function $N(n, r)$ exists, but currently we do not have a formula for one.

As an immediate consequence of the decidability of the near-unanimity problem and the near-unanimity obstacle theorem, the natural duality problem for finite algebras that generate a congruence distributive variety is also decidable. However, it is still open whether the natural duality problem in general is decidable. A very interesting group of open problem is related to the *constraint satisfaction problem*, which we do not define here and refer the reader to [3] for details. It is proved in [4] that if a set Γ of relations on a set admits a compatible near-unanimity operation, then the corresponding constraint satisfaction problem $\mathrm{CSP}(\Gamma)$ is solvable in polynomial time. Therefore, it is natural to ask the *near-unanimity problem for relations*:

Problem. *Given a finite set Γ of relations on A, decide whether there exists a near-unanimity operation on A that is compatible with each member of Γ.*

Currently we do not know the decidability of this problem, even in the light of our result. We know that if a clone has a near-unanimity operation, then both the clone and its dual relational clone are finitely generated. Inspired by this, we ask the following:

Problem. *Given a finite set of operations and a finite set of relations on the same underlying set, decide if the functional and relational clones they generate are duals of each other.*

Proof

Let ω and ω^+ be the set of all finite and countable cardinals, respectively. For a nonempty set A we denote by $\mathcal{O}_A$ the set of all operations on A. In general we do not assume that the underlying set A is finite. For $\mathcal{F} \subseteq \mathcal{O}_A$ and $n \in \omega$ put $\mathcal{F}^{(n)} = \mathcal{F} \cap A^{A^n}$, which is the set of all n-ary operations contained in $\mathcal{F}$. Binary operations will play a crucial role in our arguments, therefore we put $\mathcal{B}_A = \mathcal{O}_A^{(2)}$. The clone generated by a set $\mathcal{F} \subseteq \mathcal{O}_A$ will be denoted by $\langle \mathcal{F} \rangle$. All indices in this chapter start from zero.

An operation $f \in \mathcal{O}_A^{(n)}$ is a *near-unanimity operation* if

$$f(y, x, \dots, x) = f(x, y, x, \dots, x) = \dots = f(x, \dots, x, y) = x$$

for all $x, y \in A$. It is customary to assume that $n \geq 3$, but we will not make this restriction to avoid considering special cases in some of our arguments. However, this does not weaken our results, because no operation of arity less than three can satisfy this definition whenever the underlying set has at least two elements. The problem of deciding whether a finite algebra has a near-unanimity term operation is called the *near-unanimity problem*.

Instead of working with operations and their composition, we introduce an equivalence relation on the set of operations in such a way that

(1) the near-unanimity operations form an equivalence class of the relation,
(2) a new notion of composition can be introduced on the equivalence classes, and
(3) it is possible to algorithmically compute the closure of equivalence classes under this new notion of composition.

We start the proof with the study of the binary operations that arise as $f(x, \dots, x, y, x, \dots, x)$ from another operation $f \in \mathcal{O}_A$.

Definition 1. For $f \in \mathcal{O}_A^{(n)}$ and $i \in \omega$, the ith *polymer* of f is $f|_i \in \mathcal{B}_A$ defined as

$$f|_i(x, y) = \begin{cases} f(x, \dots, x, \overset{i\text{th}}{\smile}\!\!\!\!\!\!y, x, \dots, x) & \text{if } i < n, \\ f(x, \dots, x) & \text{if } i \geq n, \end{cases}$$

where y occurs at the ith coordinate of f in the first case. The collection of polymers of f together with their multiplicities is the *characteristic function* of f, which is formally defined as the map $\chi_f : \mathcal{B}_A \to \omega^+$ where

$$\chi_f(b) = |\{\, i \in \omega : f|_i = b \,\}| .$$

By the set of characteristic functions on a nonempty set A we mean the set $\mathcal{X}_A = \{\, \chi_f : f \in \mathcal{O}_A \,\}$. Note that not every mapping of $\mathcal{B}_A$ to ω^+ is a characteristic function of some operation. In the following lemma we characterize the ones that are.

Lemma 2. *A mapping $\chi : \mathcal{B}_A \to \omega^+$ is a characteristic function of some operation if and only if*

(1) *there exists a unique element $b \in \mathcal{B}_A$ such that $\chi(b) = \omega$,*
(2) *there are only finitely many $c \in \mathcal{B}_A$ such that $\chi(c) \neq 0$, and*
(3) *$c(x, x) \approx b(x, y)$ whenever $\chi(c) \neq 0$ and $\chi(b) = \omega$.*

Proof. To show that the given list of conditions are necessary, take an arbitrary operation $f \in \mathcal{O}_A^{(n)}$. Put $b = f|_n$. By Definition 1, $b(x, y) \approx f(x, \dots, x)$ and $f|_i = b$ for all $i \geq n$, which proves that $\chi(b) = \omega$. Moreover, for every $c \in \mathcal{B}_A$ other than b, $\chi(c) = |\{ i < n : f|_i = c \}|$ is finite, proving items (1) and (2). Finally, if $\chi(c) \neq 0$, then $c = f|_i$ for some $i \in \omega$, and $c(x,x) \approx f(x,\dots,x) \approx b(x,y)$.

To show the other direction, take a mapping $\chi : \mathcal{B}_A \to \omega^+$ satisfying items (1)–(3). Let $b \in \mathcal{B}_A$ be the unique element for which $\chi(b) = \omega$, and put $C = \{ c \in \mathcal{B}_A : \chi(c) \notin \{0, \omega\} \}$. By conditions (1) and (2), the set C is finite, and $n = \sum_{c \in C} \chi(c)$ is a finite number. Consequently, we can choose a finite list $\xi_0, \dots, \xi_{n-1} \in \mathcal{B}_A$ of elements such that $\{\xi_0, \dots, \xi_{n-1}\} = C$ and $\chi(c) = |\{ i < n : \xi_i = c \}|$ for all $c \in C$. Because of condition (3), there exists an operation $f \in \mathcal{O}_A^{(n+3)}$ that satisfies the following list of identities:

$$
\begin{aligned}
f(y,x,x,\dots,x,x,x,x,x) &\approx \xi_0(x,y),\\
f(x,y,x,\dots,x,x,x,x,x) &\approx \xi_1(x,y),\\
&\vdots\\
f(x,x,x,\dots,x,y,x,x,x) &\approx \xi_{n-1}(x,y),\\
f(x,x,x,\dots,x,x,y,x,x) &\approx b(x,y),\\
f(x,x,x,\dots,x,x,x,y,x) &\approx b(x,y),\\
f(x,x,x,\dots,x,x,x,x,y) &\approx b(x,y),\\
f(x,x,x,\dots,x,x,x,x,x) &\approx b(x,y).
\end{aligned}
$$

Clearly, $f|_i = \xi_i$ for all $i < n$, and $f|_n = f|_{n+1} = f|_{n+2} = f|_{n+3} = \cdots = b$. Therefore, $\chi_f = \chi$, which concludes the proof. □

We leave it to the reader to prove the following result that characterizes near-unanimity operations by their characteristic functions.

Lemma 3. *$f \in \mathcal{O}_A$ is a near-unanimity operation if and only if $\chi_f = \chi_{\mathrm{nu}}$ where $\chi_{\mathrm{nu}} \in \mathcal{X}_A$ is defined as*

$$
\chi_{\mathrm{nu}}(b) = \begin{cases} \omega & \text{if } b(x,y) \approx x, \\ 0 & \text{otherwise.} \end{cases}
$$

Given a set $\mathcal{G} \subseteq \mathcal{O}_A$ of operations, we define $\mathsf{X}(\mathcal{G}) = \{ \chi_f : f \in \mathcal{G} \}$. By the last lemma, the kernel of the operator $f \mapsto \chi_f$ satisfies our goal (1) stated at the beginning of the chapter. To establish goal (2), we introduce the notions of composition for operations and characteristic functions, and consequently show that they correspond to one another under taking the characteristic functions of the operations. If for a set $\mathcal{G}$ of operations we can show that the corresponding set $\{ \chi_g : g \in \mathcal{G} \}$ of characteristic functions is closed under this new notion of composition and does not include χ_{nu}, then we will be able to conclude that $\langle \mathcal{G} \rangle$ does not contain a near-unanimity operation, even if $\mathcal{G}$ is not a clone. First, we need the following definition.

Definition 4. By an *extension* of $g \in \mathcal{O}_A^{(n)}$ we mean an operation $g' \in \mathcal{O}_A^{(m)}$ satisfying

$$
g'(x_0, \dots, x_{m-1}) \approx g(x_{\sigma(0)}, \dots, x_{\sigma(n-1)}),
$$

where σ is an arbitrary injection of $\{0,\dots,n-1\}$ into $\{0,\dots,m-1\}$. By a *composition* of $f \in \mathcal{O}_A^{(n)}$ *with extensions* of $g_0,\dots,g_{n-1} \in \mathcal{O}_A$ we mean an operation of the form $f(g'_0,\dots,g'_{n-1})$ where $g'_0,\dots,g'_{n-1} \in \mathcal{O}_A^{(m)}$ are extensions of $g_0,\dots,g_{n-1}$, respectively, and are of the same arity m.

Clearly, the extensions of g are exactly the operations that can be obtained from g by permuting the variables and introducing dummy variables. As an example, all projections are extensions of the unary projection. It is easy to see that if g' is an extension of g, then $\chi_{g'} = \chi_g$.

The full meaning of the following definition well be revealed in the proof of Lemma 6, but first we motivate it by a simple example. Take operations $f \in \mathcal{O}_A^{(2)}$ and $g_0, g_1 \in \mathcal{O}_A^{(m)}$. We would like to describe the characteristic function of $f(g_0,g_1)$ via the characteristic functions of f, g_0 and g_1. Clearly, the ith polymer of $f(g_0,g_1)$ is $f(g_0|_i, g_1|_i)$, which shows that $\chi_{f(g_0,g_1)}$ depends not only on χ_f but also on f. Furthermore, if g'_1 is an m-ary extensions of g_1, then $\chi_{g_1} = \chi_{g'_1}$, but in general $g_1|_i \neq g'_1|_i$, and therefore $\chi_{f(g_0,g_1)} \neq \chi_{f(g_0,g'_1)}$. This shows that besides χ_{g_0} and χ_{g_1} we also need to know which "variables" of χ_{g_0} correspond to the "variables" of χ_{g_1}. What we need is an assignment, denoted as a map μ in the following definition, that with multiplicities assigns the polymers of g_0 to that of g_1.

Definition 5. We say that $\chi \in \mathcal{X}_A$ is a *composition* of $f \in \mathcal{O}_A^{(n)}$ with $\chi_0,\dots,\chi_{n-1} \in \mathcal{X}_A$ if there exists a mapping $\mu : (\mathcal{B}_A)^n \to \omega^+$ such that

$$\chi(c) = \sum_{\bar{b} \in (\mathcal{B}_A)^n,\ f(\bar{b}) = c} \mu(\bar{b})$$

and

$$\chi_i(c) = \sum_{\bar{b} \in (\mathcal{B}_A)^n,\ b_i = c} \mu(\bar{b})$$

for all $c \in \mathcal{B}_A$ and $i < n$.

We introduce the following operators on $\mathcal{O}_A$ and $\mathcal{X}_A$. Given $\mathcal{F}, \mathcal{G} \subseteq \mathcal{O}_A$, we denote by $\mathsf{C}_{\mathcal{F}}(\mathcal{G})$ the set of all possible compositions of operations $f \in \mathcal{F}^{(n)}$ with extensions of $g_0,\dots,g_{n-1} \in \mathcal{G}$. We will use the same symbol for the analogous operator for characteristic functions: given $\mathcal{F} \subseteq \mathcal{O}_A$ and $\mathcal{U} \subseteq \mathcal{X}_A$, we denote by $\mathsf{C}_{\mathcal{F}}(\mathcal{U})$ the set of all possible compositions of operations $f \in \mathcal{F}^{(n)}$, for some $n \in \omega$, with characteristic functions $\chi_0,\dots,\chi_{n-1} \in \mathcal{U}$.

Lemma 6. $\mathsf{X}\mathsf{C}_{\mathcal{F}}(\mathcal{G}) = \mathsf{C}_{\mathcal{F}}\mathsf{X}(\mathcal{G})$ *for all* $\mathcal{F}, \mathcal{G} \subseteq \mathcal{O}_A$.

Proof. To prove the inclusion $\subseteq$, take $f \in \mathcal{F}^{(n)}$ and $g_0,\dots,g_{n-1} \in \mathcal{G}$, let $g'_0,\dots,g'_{n-1} \in \mathcal{O}_A^{(m)}$ be extensions of $g_0,\dots,g_{n-1}$, respectively, of the same arity $m \in \omega$, and put $h = f(g'_0,\dots,g'_{n-1}) \in \mathcal{O}_A^{(m)}$. We need to show that χ_h is a composition of f with $\chi_{g_0},\dots,\chi_{g_{n-1}}$. Define $\mu : (\mathcal{B}_A)^n \to \omega^+$ as

$$\mu(\bar{b}) = \left|\left\{\, i \in \omega : \langle g'_0|_i, \dots, g'_{n-1}|_i \rangle = \bar{b} \,\right\}\right|,$$

which describes how many times the tuple $\bar b \in (\mathcal{B}_A)^n$ of binary operations appear as the polymers of $g'_0, \dots, g'_{n-1}$ at the same coordinate i.

We check Definition 5 now. For each element $c \in \mathcal{B}_A$,

$$\sum_{\bar b \in (\mathcal{B}_A)^n,\ f(\bar b)=c} \mu(\bar b) = \big|\{\, i \in \omega : f(g'_0|_i, \dots, g'_{n-1}|_i) = c \,\}\big|$$
$$= \big|\{\, i \in \omega : h|_i = c \,\}\big| = \chi_h(c).$$

On the other hand, for each $j < n$ and $c \in \mathcal{B}_A$,

$$\sum_{\bar b \in (\mathcal{B}_A)^n,\ b_j=c} \mu(\bar b) = \big|\{\, i \in \omega : g'_j|_i = c \,\}\big| = \chi_{g'_j}(c).$$

This shows that χ_h is a composition of f with $\chi_{g'_0}, \dots, \chi_{g'_{n-1}}$. Moreover, since g'_j is an extension of g_j, $\chi_{g_j} = \chi_{g'_j}$ for all $j < n$. This completes the proof of $\mathsf{XC}_{\mathcal{F}}(\mathcal{G}) \subseteq \mathsf{C}_{\mathcal{F}}\mathsf{X}(\mathcal{G})$.

To prove the other inclusion, take an arbitrary $\chi \in \mathsf{C}_{\mathcal{F}}\mathsf{X}(\mathcal{G})$. Then there exist $f \in \mathcal{F}^{(n)}$, operations $g_0, \dots, g_{n-1} \in \mathcal{G}$ of arities $m_0, \dots, m_{n-1}$, respectively, and $\mu : (\mathcal{B}_A)^n \to \omega^+$ such that

$$\chi(c) = \sum_{\bar b \in (\mathcal{B}_A)^n,\ f(\bar b)=c} \mu(\bar b) \tag{1}$$

and

$$\chi_{g_j}(c) = \sum_{\bar b \in (\mathcal{B}_A)^n,\ b_j=c} \mu(\bar b) \tag{2}$$

for all $c \in \mathcal{B}_A$ and $j < n$. We will argue that χ is the characteristic function of a composition of f with extensions of $g_0, \dots, g_{n-1}$.

Using equation (1) we obtain

$$\sum_{\bar b \in (\mathcal{B}_A)^n} \mu(\bar b) = \sum_{c \in \mathcal{B}_A} \chi(c) = \omega,$$

where the second equality holds because χ is a characteristic function. Consequently, we can choose a mapping $\xi : \omega \to (\mathcal{B}_A)^n$ such that

$$\mu(\bar b) = \big|\{\, i \in \omega : \xi(i) = \bar b \,\}\big|$$

for all $\bar b \in (\mathcal{B}_A)^n$. Now, using equation (2), we get that

$$\big|\{\, i \in \omega : g_j|_i = c \,\}\big| = \chi_{g_j}(c) = \sum_{\bar b \in (\mathcal{B}_A)^n,\ b_j=c} \mu(\bar b) = \big|\{\, i \in \omega : \xi(i)_j = c \,\}\big|$$

for all $j < n$ and $c \in \mathcal{B}_A$. The cardinalities of the two sets on the two sides are equal, therefore, for every $j < n$ we can choose a permutation $\sigma_j : \omega \to \omega$ such that

$$g_j|_i = \xi(\sigma_j(i))_j$$

for all $i \in \omega$. Put $m = \max\{\, \sigma_j(i) : j < n, i < m_j \,\}$. Now, for all $j < n$, the restriction of σ_j to the set $\{0, \dots, m_j - 1\}$ is an injection into the set $\{0, \dots, m-1\}$. Define the operations $g'_0, \dots, g'_{n-1} \in \mathcal{O}_A^{(m)}$ as

$$g'_j(x_0, \dots, x_{m-1}) \approx g_j(x_{\sigma_j(0)}, \dots, x_{\sigma_j(m_j-1)}).$$

Clearly, each g'_j is an extension of g_j. To complete the proof, we need to show that the characteristic function of $f(g'_0, \dots, g'_{n-1})$ equals χ.

Observe that

$$g'_j|_i = \begin{cases} g_j|_{\sigma_j^{-1}(i)} & \text{if } \sigma_j^{-1}(i) < m_j, \\ g_j(x, \dots, x) & \text{otherwise.} \end{cases}$$

As a result, $g'_j|_i = g_j|_{\sigma_j^{-1}(i)}$ for all $i \in \omega$, and therefore

$$g'_j|_i = g_j|_{\sigma_j^{-1}(i)} = \xi(\sigma_j\sigma_j^{-1}(i))_j = \xi(i)_j$$

for all $i \in \omega$ and $j < n$. Then, for an arbitrary element $c \in \mathcal{B}_A$,

$$\begin{aligned} \chi_{f(g'_0,\dots,g'_{n-1})}(c) &= \bigl|\{\, i \in \omega : f(g'_0, \dots, g'_{n-1})|_i = c \,\}\bigr| \\ &= \bigl|\{\, i \in \omega : f(g'_0|_i, \dots, g'_{n-1}|_i) = c \,\}\bigr| \\ &= \bigl|\{\, i \in \omega : f(\xi(i)_0, \dots, \xi(i)_{n-1}) = c \,\}\bigr| \\ &= \bigl|\{\, i \in \omega : f(\bar{b}) = c \text{ where } \bar{b} = \xi(i) \,\}\bigr| \\ &= \sum_{\bar{b} \in (\mathcal{B}_A)^n,\ f(\bar{b}) = c} \mu(\bar{b}) \\ &= \chi(c). \end{aligned}$$

□

The following lemma turns the near unanimity problem into a problem about characteristic functions. We will use the power notation for the composition operator. For $\mathcal{F}, \mathcal{G} \subseteq \mathcal{O}_A$ we define $\mathsf{C}^0_{\mathcal{F}}(\mathcal{G}) = \mathcal{G}$, and $\mathsf{C}^{n+1}_{\mathcal{F}}(\mathcal{G}) = \mathsf{C}_{\mathcal{F}}\mathsf{C}^n_{\mathcal{F}}(\mathcal{G})$ for all $n \in \omega$. We use the same power notation for the composition of characteristic functions, as well.

Lemma 7. *Let $\mathcal{F} \subseteq \mathcal{O}_A$ and $\mathcal{G} \subseteq \langle \mathcal{F} \rangle$, and assume that $\mathcal{G}$ contains an idempotent operation. Then $\langle \mathcal{F} \rangle$ contains a near-unanimity operation if and only if $\chi_{\text{nu}} \in \bigcup_{n\in\omega} \mathsf{C}^n_{\mathcal{F}}\mathsf{X}(\mathcal{G})$.*

Proof. By Lemma 6, $\bigcup_{n\in\omega} \mathsf{C}^n_{\mathcal{F}}\mathsf{X}(\mathcal{G}) = \mathsf{X}\left(\bigcup_{n\in\omega} \mathsf{C}^n_{\mathcal{F}}(\mathcal{G})\right)$. Consequently, by Lemma 3, it is enough to show that $\langle \mathcal{F} \rangle$ contains a near-unanimity operation if and only if $\bigcup_{n\in\omega} \mathsf{C}^n_{\mathcal{F}}(\mathcal{G})$ does. One direction is trivial because $\bigcup_{n\in\omega} \mathsf{C}^n_{\mathcal{F}}(\mathcal{G}) \subseteq \langle \mathcal{F} \rangle$. For the other direction assume that $f \in \langle \mathcal{F} \rangle^{(k)}$ is a near-unanimity operation and $g \in \mathcal{G}^{(m)}$ is an arbitrary idempotent operation. We define $h \in \langle \mathcal{F} \rangle^{(km)}$ as

$$h(x_0, \dots, x_{km-1}) = f(g(x_0, \dots, x_{m-1}), \dots, g(x_{km-m}, \dots, x_{km-1})).$$

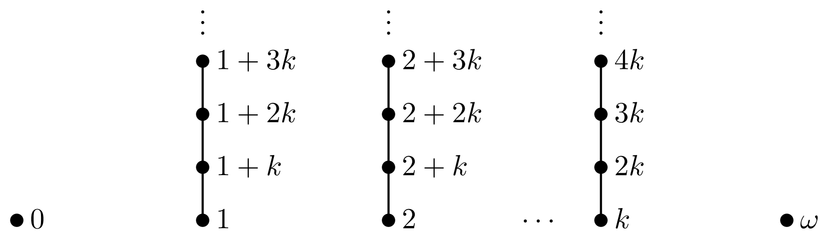
Clearly, h is a near-unanimity operation, and $h \in \bigcup_{n\in\omega} \mathsf{C}^n_{\mathcal{F}}(\mathcal{G})$. □

If $\mathcal{G}$ is the set of all projections on a set A and $\mathcal{F} \subseteq \mathcal{O}_A$, then $\bigcup_{n\in\omega} \mathsf{C}^n_{\mathcal{F}}(\mathcal{G}) = \langle \mathcal{F} \rangle$, and $\mathsf{X}(\mathcal{G}) = \{\chi_{\mathrm{id}}\}$, where χ_{id} is defined as

$$\chi_{\mathrm{id}}(b) = \begin{cases} \omega & \text{if } b(x,y) \approx x, \\ 1 & \text{if } b(x,y) \approx y, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by the previous lemma, $\langle \mathcal{F} \rangle$ contains a near-unanimity operation if and only if $\chi_{\mathrm{nu}} \in \bigcup_{n\in\omega} \mathsf{C}^n_{\mathcal{F}}(\{\chi_{\mathrm{id}}\})$. However, this condition does not seem to be easier to check than the original one. We overcome this problem by carefully choosing $\mathcal{G}$ so that the latter condition can be effectively tested.

Definition 8. For an integer $k \geq 1$ we define a partial order $\sqsubseteq_k$ on ω^+ as follows:



Acting coordinate-wise, this defines a partial order on $\mathcal{X}_A$. For a set $\mathcal{U} \subseteq \mathcal{X}_A$ denote by $\mathsf{F}_k(\mathcal{U})$ the order filter generated by $\mathcal{U}$ in $\mathcal{X}_A$, that is,

$$\mathsf{F}_k(\mathcal{U}) = \{\, \chi' \in \mathcal{X}_A : (\exists \chi \in \mathcal{U})(\forall b \in \mathcal{B}_A)(\chi(b) \sqsubseteq_k \chi'(b)) \,\}.$$

Recall that a partially ordered set (or simply poset) is called *well-ordered*, if it has no infinite anti-chains and satisfies the descending chain condition, i.e., contains no strictly decreasing sequence of elements. Clearly, $\langle \omega^+; \sqsubseteq_k \rangle$ is well-ordered. It is known that subposets and finite products of well-ordered posets are well-ordered (these are elementary facts, see e.g. [5]). Moreover, the set of order filters of a well-ordered poset under the inclusion order satisfies the ascending chain condition. Consequently, provided that A is finite, $\langle \mathcal{X}_A; \sqsubseteq_k \rangle$ is well-ordered and has no strictly increasing sequence of order filters. From now on A is assumed to be finite.

Lemma 9. *Let $k \geq 1$, $\mathcal{F} \subseteq \mathcal{O}_A$ and $\mathcal{U} \subseteq \mathcal{X}_A$. Then $\mathsf{F}_k\mathsf{C}_{\mathcal{F}}(\mathcal{U}) \subseteq \mathsf{C}_{\mathcal{F}}\mathsf{F}_k(\mathcal{U})$. Consequently, $\mathsf{C}_{\mathcal{F}}\mathsf{F}_k(\mathcal{U})$ is an order filter.*

Proof. Take arbitrary characteristic functions $\chi \in \mathsf{C}_{\mathcal{F}}(\mathcal{U})$ and $\chi' \in \mathcal{X}_A$ such that $\chi \sqsubseteq_k \chi'$. Thus χ is a composition of an operation $f \in \mathcal{F}^{(n)}$ and characteristic functions $\chi_0, \dots, \chi_{n-1} \in \mathcal{U}$. By Definition 5, there exists a map $\mu : (\mathcal{B}_A)^n \to \omega^+$ such that

$$\chi(c) = \sum_{\bar{b} \in (\mathcal{B}_A)^n,\ f(\bar{b}) = c} \mu(\bar{b}) \tag{3}$$

and

$$\chi_i(c) = \sum_{\bar{b} \in (\mathcal{B}_A)^n,\ b_i = c} \mu(\bar{b}) \tag{4}$$

for all $c \in \mathcal{B}_A$ and $i < n$. Let D be the set of binary operations $d \in \mathcal{B}_A$ where $\chi(d) \neq \chi'(d)$. Since neither 0 nor ω is comparable to any other element under $\sqsubseteq_k$, for all $d \in D$, $\chi(d) \notin \{0, \omega\}$ and $\chi'(d) - \chi(d)$ equals to a positive multiple of k. Using equation (3), for each $d \in D$ we can choose an n-tuple $\bar{b}_d \in (\mathcal{B}_A)^n$ such that $f(\bar{b}_d) = d$ and $\mu(\bar{b}_d) \notin \{0, \omega\}$. Define $\mu' : (\mathcal{B}_A)^n \to \omega^+$ as

$$\mu'(\bar{b}) = \begin{cases} \mu(\bar{b}) + \chi'(d) - \chi(d) & \text{if } \bar{b} = \bar{b}_d \text{ for some } d \in D, \\ \mu(\bar{b}) & \text{otherwise.} \end{cases}$$

Clearly, $\mu(\bar{b}) \sqsubseteq_k \mu'(\bar{b})$ for all $\bar{b} \in (\mathcal{B}_A)^n$. Then by equation (4), $\chi_i \sqsubseteq_k \chi'_i$ for all $i < n$ where $\chi'_i : \mathcal{B}_A \to \omega^+$ is defined as

$$\chi'_i(c) = \sum_{\bar{b} \in (\mathcal{B}_A)^n,\ b_i = c} \mu'(\bar{b})$$

for all $c \in \mathcal{B}_A$. On the other hand, by the choice of μ',

$$\chi'(c) = \sum_{\bar{b} \in (\mathcal{B}_A)^n,\ f(\bar{b}) = c} \mu'(\bar{b})$$

for all $c \in \mathcal{B}_A$. This proves that χ' is a composition of f and the characteristic functions $\chi'_0, \dots, \chi'_{n-1} \in \mathsf{F}_k(\mathcal{U})$ via the map μ'.

To prove the second assertion of the lemma, consider the containments $\mathsf{F}_k\mathsf{C}_{\mathcal{F}}\mathsf{F}_k(\mathcal{U}) \subseteq \mathsf{C}_{\mathcal{F}}\mathsf{F}_k\mathsf{F}_k(\mathcal{U}) = \mathsf{C}_{\mathcal{F}}\mathsf{F}_k(\mathcal{U}) \subseteq \mathsf{F}_k\mathsf{C}_{\mathcal{F}}\mathsf{F}_k(\mathcal{U})$ showing that $\mathsf{C}_{\mathcal{F}}\mathsf{F}_k(\mathcal{U})$ is an order filter. □

Lemma 10. *Let $k \geq 1$, and let A, $\mathcal{F} \subseteq \mathcal{O}_A$ and $\mathcal{U} \subseteq \mathcal{X}_A$ be finite sets. Then the minimal elements of $\langle \mathsf{C}_{\mathcal{F}}\mathsf{F}_k(\mathcal{U}); \sqsubseteq_k \rangle$ can be effectively computed.*

Proof. Choose an arbitrary minimal element $\chi \in \mathsf{C}_{\mathcal{F}}\mathsf{F}_k(\mathcal{U})$. Then χ is a composition of an n-ary operation $f \in \mathcal{F}^{(n)}$ with some characteristic functions $\chi_0, \dots, \chi_{n-1} \in \mathsf{F}_k(\mathcal{U})$ via a mapping $\mu : (\mathcal{B}_A)^n \to \omega^+$. Observe in Definition 5 that f and μ uniquely determine χ and $\chi_0, \dots, \chi_{n-1}$ via the defining equations

$$\chi(c) = \sum_{\bar{b} \in (\mathcal{B}_A)^n,\ f(\bar{b}) = c} \mu(\bar{b}) \tag{5}$$

and

$$\chi_i(c) = \sum_{\bar{b} \in (\mathcal{B}_A)^n,\ b_i = c} \mu(\bar{b}). \tag{6}$$

Since A is finite, $(\mathcal{B}_A)^n$ is finite, and consequently the poset $\langle (\omega^+)^{(\mathcal{B}_A)^n}; \sqsubseteq_k \rangle$ is well ordered. Clearly, μ is an element of this poset, so we can assume that μ is minimal in this poset among all representations of χ.

By the finiteness of A and $\mathcal{U}$,

$$m = \max\big(\{k\} \cup \{\, \chi'(b) : \chi' \in \mathcal{U}, b \in \mathcal{B}_A \text{ and } \chi'(b) \neq \omega \,\}\big)$$

is a (finite) natural number that depends only on k, A and $\mathcal{U}$. We claim that $\mu(\bar{b}) \in \{0, \dots, m, \omega\}$ for all $\bar{b} \in (\mathcal{B}_A)^n$, which is enough to conclude our proof because then only finitely many operations $f \in \mathcal{F}$ and finitely many mappings $\mu : (\mathcal{B}_A)^n \to \{0, \dots, m, \omega\}$ need to be considered to find all minimal elements of $\mathsf{C}_{\mathcal{F}}\mathsf{F}_k(\mathcal{U})$.

To get a contradiction, assume that $\mu(\bar{c}) > m$ and $\mu(\bar{c}) \neq \omega$ for some tuple $\bar{c} \in (\mathcal{B}_A)^n$. Define $\mu' : (\mathcal{B}_A)^n \to \omega^+$ as

$$\mu'(\bar{b}) = \begin{cases} \mu(\bar{b}) & \text{if } \bar{b} \neq \bar{c}, \\ \mu(\bar{b}) - k & \text{if } \bar{b} = \bar{c}, \end{cases}$$

and define χ' and $\chi'_0, \dots, \chi'_{n-1}$ using the defining equations (5) and (6) for μ', respectively. Observe that $\mu'(\bar{c}) = \mu(\bar{c}) - k > m - k \geq 0$.

First we argue that $\chi'_i \in \mathsf{F}_k(\mathcal{U})$ for all $i = 0, \dots, n-1$. Clearly, by equation (6), $\chi_i(b) = \chi'_i(b)$ for all $b \neq c_i$. Moreover, either $\chi'_i(c_i) = \chi_i(c_i) = \omega$ or $\chi'_i(c_i) = \chi_i(c_i) - k$. In the former case, $\chi'_i = \chi_i \in \mathsf{F}_k(\mathcal{U})$. In the latter case, $\chi'_i(c_i) = \chi_i(c_i) - k \geq \mu(\bar{c}) - k > m - k \geq 0$, where the first inequality holds by equation (6). Therefore, χ'_i satisfies the conditions of Lemma 2, so $\chi'_i \in \mathcal{X}_A$. Since $\chi_i \in \mathsf{F}_k(\mathcal{U})$, there exists a characteristic function $\chi''_i \in \mathcal{U}$ so that $\chi''_i \sqsubseteq_k \chi_i$. By the choice of m, $\chi''_i(c_i) \leq m < \mu(\bar{c}) \leq \chi_i(c_i)$, consequently $\chi''_i(c_i) \leq \chi_i(c_i) - k$. This proves that $\chi''_i \sqsubseteq_k \chi'_i$. As a result, $\chi'_i \in \mathsf{F}_k(\mathcal{U})$.

Analogously, $\chi'(d) = \chi(d)$ for all $d \neq f(\bar{c})$, and either $\chi'(f(\bar{c})) = \chi(f(\bar{c})) = \omega$ or $\chi'(f(\bar{c})) = \chi(f(\bar{c})) - k > m - k \geq 0$. Consequently, $\chi' \in \mathcal{X}_A$ by Lemma 2, and $\chi' \sqsubseteq_k \chi$. Since $\chi'_0, \dots, \chi'_{n-1} \in \mathsf{F}_k(\mathcal{U})$, we get that $\chi' \in \mathsf{C}_{\mathcal{F}}\mathsf{F}_k(\mathcal{U})$. From the minimality of χ we see that $\chi' = \chi$. But then μ' contradicts the minimality of μ, which concludes the proof. $\square$

Lemma 11. *Let $k \geq 1$, and let A, $\mathcal{F} \subseteq \mathcal{O}_A$ and $\mathcal{U} \subseteq \mathcal{X}_A$ be finite sets. Then $\bigcup_{n\in\omega} \mathsf{C}^n_{\mathcal{F}}\mathsf{F}_k(\mathcal{U})$ is an order filter with respect to $\sqsubseteq_k$, and its minimal elements can be effectively computed.*

Proof. For every $m \in \omega$ define $\mathcal{U}_m = \bigcup_{n\leq m} \mathsf{C}^n_{\mathcal{F}}\mathsf{F}_k(\mathcal{U})$, where $\mathcal{U}_0 = \mathsf{F}_k(\mathcal{U})$. For each $m \in \omega$, $\mathcal{U}_m$ is an order filter in $\langle \mathcal{X}_A; \sqsubseteq_k \rangle$ whose minimal elements can be effectively computed by Lemmas 9 and 10. Since A is finite, $\langle \mathcal{X}_A; \sqsubseteq_k \rangle$ is well-ordered and consequently the set of all its order filters under the inclusion order satisfies the ascending chain condition. Therefore, the ascending chain $\mathcal{U}_0 \subseteq \mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \dots$ of order filters cannot be strictly increasing.

Assume that $\mathcal{U}_m = \mathcal{U}_{m+1}$ for some $m \in \omega$. This condition is equivalent to that of $\mathsf{C}^{m+1}\mathsf{F}_k(\mathcal{U}) \subseteq \bigcup_{n\leq m} \mathsf{C}^n_{\mathcal{F}}\mathsf{F}_k(\mathcal{U})$. Applying $\mathsf{C}_{\mathcal{F}}$ to both sides we get that

$$\mathsf{C}^{m+2}\mathsf{F}_k(\mathcal{U}) \subseteq \bigcup_{1\leq n\leq m+1} \mathsf{C}^n_{\mathcal{F}}\mathsf{F}_k(\mathcal{U}) \subseteq \mathcal{U}_{m+1}.$$

Consequently, $\mathcal{U}_{m+1} = \mathcal{U}_{m+2}$. By induction, we obtain that $\mathcal{U}_m = \mathcal{U}_{m+1} = \mathcal{U}_{m+2} = \dots$, as a result $\mathcal{U}_m = \bigcup_{n\in\omega} \mathsf{C}^n_{\mathcal{F}}\mathsf{F}_k(\mathcal{U})$.

This yields an algorithm to find $\bigcup_{n\in\omega} \mathsf{C}^n_{\mathcal{F}}\mathsf{F}_k(\mathcal{U})$. Calculate $\mathcal{U}_0, \mathcal{U}_1, \dots$ in order using Lemma 10. If $\mathcal{U}_m = \mathcal{U}_{m+1}$ for some $m \in \omega$, then we have found $\bigcup_{n\in\omega} \mathsf{C}^n_{\mathcal{F}}\mathsf{F}_k(\mathcal{U})$ and know its minimal elements. This condition must occur and therefore the algorithm stops, because we cannot have a strictly increasing sequence of order filters in $\langle \mathcal{X}_A; \sqsubseteq_k\rangle$. $\square$

The previous lemma shows that the minimal elements of the infinite union $\bigcup_{n\in\omega} \mathsf{C}^n_{\mathcal{F}}\mathsf{X}(\mathcal{G})$ of Lemma 7 can be effectively calculated provided that $\mathsf{X}(\mathcal{G})$ forms an order filter in $\langle \mathcal{X}_A; \sqsubseteq_k\rangle$ for some $k \geq 1$. We will argue that such integer k and set $\mathcal{G} \subseteq \langle \mathcal{F}\rangle$ can be found if $\langle \mathcal{F}\rangle$ contains a near-unanimity operation. We need the following definition.

Definition 12. Let $k \in \omega$ and $f \in \mathcal{O}^{(n)}_A$. We call f a *k-nu operation* if $k \leq n$ and

$$\begin{gathered} f(x,\dots,x) \approx x, \\ f|_0(x,y) \approx \dots \approx f|_{k-1}(x,y) \text{ and} \\ f|_k(x,y) \approx \dots \approx f|_{n-1}(x,y) \approx x. \end{gathered}$$

This concept is the generalization of that of near-unanimity and weak near-unanimity operations. The 0-nu operations are precisely the near-unanimity operations, while the k-nu operations of arity k are called weak near-unanimity operations.

Lemma 13. *If a clone on an m-element set contains a near-unanimity operation, then it contains a 2-nu operation of arity at most $2 + m^{m^2}$.*

To prove this lemma, we need the following theorem.

Theorem 14 (L. Lovász [6])**.** *Let n, k be natural numbers such that $2 \leq 2k \leq n$, and $G_{n,k}$ be the graph on the set of all k-element subsets of an n-element set with the disjointness relation. Then the chromatic number of $G_{n,k}$ is $n - 2k + 2$.* $\square$

Proof of Lemma 13. Let $\mathcal{C}$ be a clone and $f \in \mathcal{C}$ be a near-unanimity operation of arity n. If $n \leq 1 + m^{m^2}$, then we are done as f is a 2-nu operation. Otherwise $n - m^{m^2} \geq 2$. Put

$$k = \left\lfloor \frac{n - m^{m^2} + 1}{2} \right\rfloor.$$

By the choice of k, we have $n - m^{m^2} \leq 2k \leq n - m^{m^2} + 1$, from which it follows that $1 + m^{m^2} \leq n - 2k + 2 \leq 2 + m^{m^2}$ and $2 \leq 2k \leq n$.

We color each k-element subset $I \subseteq \{0, \dots, n-1\}$ by the binary operation $f|_I$ defined as

$$f|_I(x,y) = f(u_0,\dots,u_{n-1}) \quad \text{where} \quad u_i = \begin{cases} x & \text{if } i \notin I, \\ y & \text{if } i \in I. \end{cases}$$

There are m^{m^2} binary operations on an m-element set, thus we colored the graph $G_{n,k}$ with m^{m^2} colors. Since the chromatic number of this graph is $n-2k+2$, by Theorem 14, and $n-2k+2 > m^{m^2}$, there must exist two disjoint k-element subsets $I, J \subset \{0,\dots,n-1\}$ for which $f|_I = f|_J$.

Choose an arbitrary bijection τ from $\{0,\dots,n-1\} \setminus (I \cup J)$ to $\{0,\dots,n-2k-1\}$. We claim that the following operation is a 2-nu operation in $\mathcal{C}$ of arity at most $2 + m^{m^2}$:

$$g(x,y,z_0,\dots,z_{n-2k-1}) = f(u_0,\dots,u_{n-1}) \quad \text{where} \quad u_i = \begin{cases} x & \text{if } i \in I, \\ y & \text{if } i \in J, \\ z_{\tau(i)} & \text{otherwise.} \end{cases}$$

Clearly, $g \in \mathcal{C}$ and its arity is $n-2k+2 \le 2+m^{m^2}$. Moreover, $g|_0 = f|_I = f|_J = g|_1$, and for all $i \ge 2$, $g|_i = f|_{\tau^{-1}(i-2)} = x$ because f was a near-unanimity operation. This proves that g is a 2-nu operation. □

Lemma 15. *Let $\mathcal{C}$ be a clone on an m-element set that contains a k-nu operation of arity $k+n$. Then $\mathcal{C}$ contains a $k^{m!}$-nu operation f of arity $k^{m!}+n$ such that*

$$f|_0(x, f|_0(x,y)) = f|_0(x,y).$$

Proof. Let A be the underlying set of $\mathcal{C}$, and $g \in \mathcal{C}$ be a k-nu operation of arity $k+n$. By induction we define a sequence $g_1, g_2, g_3, \dots \in \mathcal{C}$ of operations of arities $k+n, k^2+n, k^3+n, \dots$, respectively. Put $g_1 = g$, and for $i \ge 1$ put

$$\begin{aligned} &g_{i+1}(x_0,\dots,x_{k^{i+1}-1}, y_0,\dots,y_{n-1}) \\ &\qquad = g\big(g_i(x_0,\dots,x_{k^i-1}, y_0,\dots,y_{n-1}),\dots, \\ &\qquad\qquad g_i(x_{(k-1)k^i},\dots,x_{k^{i+1}-1}, y_0,\dots,y_{n-1}), y_0,\dots,y_{n-1}\big). \end{aligned}$$

Since g is idempotent, i.e. $g(x,\dots,x) = x$, the defined operations $g_1, g_2, \dots$ are idempotent, as well. For each element $x \in A$ define the unary operation $h_x(y) = g|_0(x,y)$. We claim that, for each $i \ge 1$ and $j \in \omega$,

$$g_i|_j(x,y) = \begin{cases} h_x^i(y) & \text{if } j < k^i, \\ x & \text{if } j \ge k^i. \end{cases}$$

This holds for g_1 by definition. Let $i \ge 1$ and $j < k^{i+1}$. Choosing $l < k$ such that $lk^i \le j < (l+1)k^i$ we get that

$$\begin{aligned} g_{i+1}|_j(x,y) &= g\big(g_i(x,\dots,x),\dots,g_i(x,\dots,x), g_i|_{j-lk^i}(x,y), \\ &\qquad g_i(x,\dots,x),\dots,g_i(x,\dots,x), x,\dots,x\big) \\ &= g|_l(x, g_i|_{j-lk^i}(x,y)) \\ &= h_x(h_x^i(y) \\ &= h_x^{i+1}(y). \end{aligned}$$

Finally, if $i \geq 1$ and $k^{i+1} \leq j < k^{i+1} + n$, then

$$\begin{aligned} g_{i+1}|_j(x,y) &= g(g_i(x,\dots,x),\dots,g_i(x,\dots,x),x,\dots,x,y,x,\dots,x)) \\ &= g|_{j-k^{i+1}+k}(x,y) \\ &= x. \end{aligned}$$

This proves that each g_i is a k^i-nu operation of arity $k^i + n$. We argue that $f = g_{m!}$ is the operation we claimed in the statement of the lemma. Indeed, since h_x is a unary operation on an m-element set, it is elementary to verify that $h_x^{m!}$ is idempotent, that is, $h_x^{m!} = h_x^{2\cdot m!}$. Then,

$$f|_0(x, f|_0(x,y)) = h_x^{m!}(h_x^{m!}(y)) = h_x^{m!}(y) = f|_0(x,y). \qquad \square$$

Lemma 16. *Let A be a finite set of size m.*

(1) *If a clone on A contains a near-unanimity operation, then it contains a $2^{m!}$-nu operation g of arity at most $2^{m!} + m^{m^2}$ that satisfies*
$$g|_0(x, g|_0(x,y)) \approx g|_0(x,y).$$
(2) *If $g \in \mathcal{O}_A$ is a $2^{m!}$-nu operation satisfying the above identity, then there exists a set $\mathcal{G} \subseteq \langle\{g\}\rangle$ such that $\mathcal{G}$ contains an idempotent operation and $\mathsf{X}(\mathcal{G}) = \mathsf{F}_{2^{m!}-1}(\{\chi_g\})$.*

Proof. The first statement follows immediately from Lemmas 13 and 15. To prove the second statement, let g be a $2^{m!}$-nu operation of arity $2^{m!} + k$ that satisfies the identity of the lemma. If g is a near-unanimity operation, then we can choose $\mathcal{G} = \{g\}$. Thus assume that g is not a near-unanimity operation. By induction, we define a sequence of operations $g_i \in \langle\{g\}\rangle$ ($i = 1, 2, \dots$) of arity $i(2^{m!} - 1) + 1 + k$, respectively. Put $g_1 = g$, and for all positive integers i define

$$\begin{aligned} (7) \quad & g_{i+1}(x_0,\dots,x_{(i+1)(2^{m!}-1)}, y_0,\dots,y_{k-1}) \\ &\qquad = g_i\big(g(x_0,\dots,x_{2^{m!}-1}, y_0,\dots,y_{k-1}), \\ &\qquad\qquad x_{2^{m!}},\dots,x_{(i+1)(2^{m!}-1)}, y_0,\dots,y_{k-1}\big). \end{aligned}$$

We claim that each g_i is a $(i(2^{m!} - 1) + 1)$-nu operation and $g_i|_0 = g|_0$. This holds trivially for g_1. We prove this by induction, so assume that the claim holds for g_i. Clearly, g_{i+1} is idempotent. If $0 \leq j < 2^{m!}$, then

$$g_{i+1}|_j(x,y) \approx g_i|_0(x, g|_j(x,y)) \approx g|_0(x, g|_0(x,y)) \approx g|_0(x,y),$$

where the first identity follows from (7), $g_i|_0 = g|_0$ by the induction assumption, $g|_j = g|_0$ since g is a $2^{m!}$-nu operation, and finally the last identity was assumed in the statement of the lemma. On the other hand, if $2^{m!} \leq j \leq (i+1)(2^{m!} - 1)$, then

$$g_{i+1}|_j(x,y) \approx g_i|_{j-(2^{m!}-1)}(x,y) \approx g|_0(x,y),$$

where the first identity holds because the first argument of g_i on the right hand side of equation (7) is $g(x,\dots,x) \approx x$, and the variable x_j is at the $j - (2^{m!} - 1)$-th argument of g_i. Finally, if $(i+1)(2^{m!} - 1) < j \leq (i+1)(2^{m!} -$

$1) + k$, i.e., we plug in y into one of the y coordinates in equation (7) and x everywhere else, then we get $g_{i+1}|_j(x, y) \approx x$, because $g|_{j-i(2^{m!}-1)}(x, y) \approx x$ and $g_i|_{j-(2^{m!}-1)}(x, y) \approx x$. This finishes the proof of the claim.

From the claim it immediately follows that

$$\chi_{g_i}(b) = \begin{cases} \omega & \text{if } b(x, y) \approx x, \\ i(2^{m!} - 1) + 1 & \text{if } b(x, y) \approx g|_0(x, y), \\ 0 & \text{otherwise,} \end{cases}$$

which is well defined, because $g|_0(x, y) \not\approx x$ since we assumed that g is not a near-unanimity operation. Now put $\mathcal{G} = \{g_1, g_2, \dots\}$. Clearly, $\mathsf{X}(\mathcal{G}) = \mathsf{F}_{2^{m!}-1}(\{\chi_g\})$. □

Theorem 17. *Given a finite set A and a finite set $\mathcal{F}$ of operations on A, it is decidable whether the clone generated by $\mathcal{F}$ contains a near-unanimity operation.*

Proof. Put $m = |A|$. First we check if $\langle \mathcal{F} \rangle$ contains a $2^{m!}$-nu operation of arity at most $2^{m!} + m^{m^2}$ that satisfies the identity of Lemma 16. If such an operation is not found, then $\langle \mathcal{F} \rangle$ cannot have a near-unanimity operation. If $g \in \langle \mathcal{F} \rangle$ is such an operation, then by the same lemma we know that there exists a set $\mathcal{G} \subseteq \langle \{g\} \rangle \subseteq \langle \mathcal{F} \rangle$ of operations such that $\mathcal{G}$ contains an idempotent operation and $\mathsf{X}(\mathcal{G}) = \mathsf{F}_{2^{m!}-1}(\{\chi_g\})$. We do not need to "compute" the set $\mathcal{G}$, in fact it is infinite. Then by Lemma 11, the minimal elements of the order filter

$$\mathcal{U} = \bigcup_{n \in \omega} \mathsf{C}^n_{\mathcal{F}} \mathsf{F}_{2^{m!}-1}(\{\chi_g\}) = \bigcup_{n \in \omega} \mathsf{C}^n_{\mathcal{F}} \mathsf{X}(\mathcal{G})$$

can be effectively computed. By Lemma 7, the clone $\langle \mathcal{F} \rangle$ contains a near-unanimity operation if and only if $\chi_{\text{nu}} \in \mathcal{U}$. But this can be easily checked if we know the minimal elements of $\mathcal{U}$. In fact, χ_{nu} is minimal in $\langle \mathcal{X}_A; \sqsubseteq_{2^{m!}-1} \rangle$, and therefore must be among the minimal elements of $\mathcal{U}$. □

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